Simple Proof That $\pi^3 < 3^\pi$

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Introduction

This document presents a very simple mathematical proof that $\pi^3 < 3^\pi$. The proof illustrates the beauty of pure mathematics, which allows a simple question to be shown in verbose manner.

Theorem 1.

$$\pi^3 < 3^\pi$$

Proposition 2. *Since the both bases are over 1, we can exponentiate both sides of the inequality by $\frac{1}{3\pi}$. So we have to prove that:*

$$\pi^{\frac{1}{\pi}} < 3^{\frac{1}{3}}$$

Limit Analysis

Consider the function

$$f(x) = x^{\frac{1}{x}}, \quad x > 0.$$

First, take the logarithm on both sides.

$$\log f(x) = \frac{1}{x} \log x \quad \text{(log is a natural logarithm)}$$

$$\therefore f(x) = e^{\frac{1}{x} \log x}$$

Then analyze the limits at the boundaries of the domain:

$$\lim_{x \to 0^+} f(x) = \lim_{X \to -\infty} e^X = 0$$

And

$$\lim_{x \to \infty} f(x) = \lim_{X \to 0^+} e^X = 1.$$

Derivative Analysis

To find the derivative of $f(x)$, use logarithmic differentiation as in Limit Analysis:

$$y = x^{\frac{1}{x}} \implies \log y = \frac{\ln x}{x}.$$

Differentiating both sides:

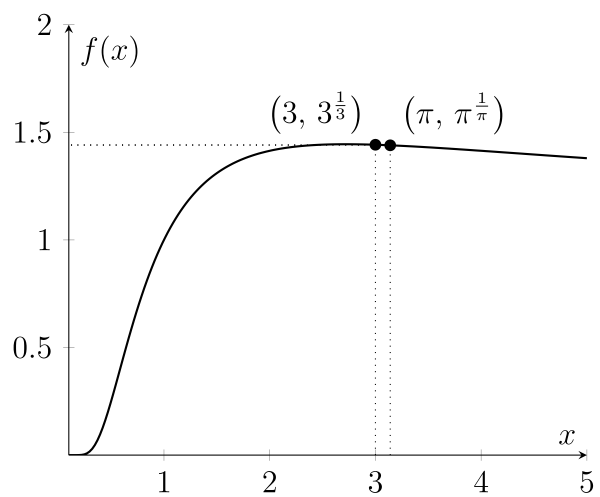
$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{1 - \log x}{x^2}.$$

Since $x > 0$,

$$\frac{1 - \log x}{x^2} = 0 \quad \text{iff.} \quad 1 - \mathrm{log\,x} = 0$$

$$\therefore x = e \quad \text{where} \quad \frac{d}{dx}f(x) = 0$$

Graph



Proof. As shown in the graph,

$$\pi^{\frac{1}{\pi}} < 3^{\frac{1}{3}}$$
$$\therefore \boxed{\pi^3 < 3^\pi}$$

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